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**ADVANCES IN
Mathematics**

Advances in Mathematics 212 (2007) 315–337

www.elsevier.com/locate/aim

General twisting of algebras [☆]

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Received 10 May 2006; accepted 10 October 2006

Available online 17 November 2006

Communicated by Michael J. Hopkins

Abstract

We introduce the concept of *pseudotwistor* (with particular cases called *twistor* and *braided twistor*) for an algebra (A, μ, u) in a monoidal category, as a morphism $T : A \otimes A \rightarrow A \otimes A$ satisfying a list of axioms ensuring that $(A, \mu \circ T, u)$ is also an algebra in the category. This concept provides a unifying framework for various deformed (or twisted) algebras from the literature, such as twisted tensor products of algebras, twisted bialgebras and algebras endowed with Fedosov products. Pseudotwistors appear also in other topics from the literature, e.g. Drinfeld's braided quantum groups and ribbon algebras. We also focus on the effect of twistors on the universal first order differential calculus, as well as on lifting twistors to braided twistors on the algebra of universal differential forms.

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Keywords: Twisted tensor product of algebras; Twisted bialgebras; Fedosov product; Braided monoidal category

[☆] Research partially supported by the EC programme LIEGRITS, RTN 2003, 505078. The first author has been also partially supported by the projects MTM2004-08125 and FQM-266 (Junta de Andalucía Research Group) and Spanish MEC-FPU grant AP2003-4340. The second and the third authors have been also partially supported by the bilateral project “New techniques in Hopf algebras and graded ring theory” of the Flemish and Romanian Ministries of Research. The second author has been also partially supported by the CEEX programme of the Romanian Ministry of Education and Research, contract nr. CEx05-D11-11/2005.

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1. Introduction

The twisted tensor product $A \otimes_R B$ of two associative algebras A and B is a certain associative algebra structure on the vector space $A \otimes B$, defined in terms of a so-called *twisting map* $R: B \otimes A \rightarrow A \otimes B$, having the property that it coincides with the usual tensor product $A \otimes B$ if R is the usual flip. This construction was proposed in [10] as a representative for the Cartesian product of noncommutative spaces. More evidence that this proposal is meaningful appeared recently in [19], where it was proved that this construction may be iterated in a natural way, and that the noncommutative $2n$ -planes defined by Connes and Dubois-Violette, cf. [11], may be written as iterated twisted tensor products of some commutative algebras. Various other applications of twisted tensor products appear in the literature, see for instance [7,31]. Note also that, as we learned from the referee, categorical analogues of twisting maps appeared earlier in the literature, under the name *distributive laws*, see for instance [2,24,29].

On the other hand, if H is a bialgebra and $\sigma: H \otimes H \rightarrow k$ is a normalized and convolution invertible left 2-cocycle, one can consider the “twisted bialgebra” ${}_{\sigma}H$, which is an associative algebra structure on H with multiplication $a * b = \sigma(a_1, b_1)a_2b_2$. This is an important and well-known construction, containing as particular case the classical twisted group rings.

Apparently, there is no relation between twisted tensor products of algebras and twisted bialgebras, except for the fact that their names suggest that they are both obtained via a process of *twisting*. However, as a consequence of the ideas developed in this paper, it will turn out that this suggestion is correct: we will find a framework in which both these constructions fit as particular cases.

Our initial aim was to relate the multiplications $\mu_{A \otimes_R B}$ of $A \otimes_R B$ and $\mu_{A \otimes B}$ of $A \otimes B$. It is easy to see that $\mu_{A \otimes_R B} = \mu_{A \otimes B} \circ T$, where $T: (A \otimes B) \otimes (A \otimes B) \rightarrow (A \otimes B) \otimes (A \otimes B)$ is a map depending on R , and the problem is to find the abstract properties satisfied by this map T , which together with the associativity of $\mu_{A \otimes B}$ imply the associativity of $\mu_{A \otimes_R B}$. We are thus led to introduce the concept of *twistor* for an algebra D , as a linear map $T: D \otimes D \rightarrow D \otimes D$ satisfying a list of axioms which imply that the new multiplication $\mu_D \circ T$ is an associative algebra structure on the vector space D (these axioms are similar to, but different from, the ones of an R -matrix for an associative algebra, a concept introduced by Borchers). It turns out that the map T affording the multiplication of $A \otimes_R B$ is such a twistor, and that various other examples of twistors may be identified in the literature, in particular the noncommutative $2n$ -plane may be regarded as a deformation of a polynomial algebra via a twistor.

But there exist in the literature many examples of deformed multiplications which are *not* afforded by twistors. For instance, the map $T(a \otimes b) = \sigma(a_1, b_1)a_2 \otimes b_2$ affording the multiplication of ${}_{\sigma}H$ is far from being a twistor. But the map $T(\omega \otimes \zeta) = \omega \otimes \zeta - (-1)^{|\omega|}d(\omega) \otimes d(\zeta)$, affording the so-called Fedosov product, is not too far, it looks like a graded analogue. We are thus led to a more general concept, called *braided twistor*, of which this T is an example. And from this concept we arrive at a much more general one, called *pseudotwistor*, which is general enough to include as example the map affording the multiplication of ${}_{\sigma}H$, as well as some other (nonrelated) situations from the literature, e.g. some examples arising in the context of Durdjević’s braided quantum groups, and the morphism $c_{A,A}^2$, where A is an algebra in a braided monoidal category with braiding c .

We also present some properties of (pseudo)twistors, e.g. we show how to lift modules and bimodules over D to the same structures over the deformed algebra, and how to extend a twistor T for an algebra D to a braided (graded) twistor \tilde{T} for the algebra of universal differential forms ΩD .

2. Preliminaries

Let k be a field, used as a base field throughout. We denote \otimes_k by \otimes , the identity id_V of an object V simply by V , and by $\tau: V \otimes W \rightarrow W \otimes V$, $\tau(v \otimes w) = w \otimes v$, the usual flip. All algebras are assumed to be associative unital k -algebras; the multiplication and unit of an algebra D are denoted by $\mu_D: D \otimes D \rightarrow D$ and respectively $u_D: k \rightarrow D$ (or simply by μ and u if there is no danger of confusion). For bialgebras and Hopf algebras we use the Sweedler-type notation $\Delta(h) = h_1 \otimes h_2$, and for categorical terminology we refer to [20,21,23]. For some proofs, we will use braiding notation, of which a detailed description may be found in [21].

We recall the twisted tensor product of algebras from [10,30,31]. If A and B are two algebras, a linear map $R: B \otimes A \rightarrow A \otimes B$ is called a *twisting map* if it satisfies the conditions

$$R(b \otimes 1) = 1 \otimes b, \quad R(1 \otimes a) = a \otimes 1, \quad \forall a \in A, b \in B, \quad (2.1)$$

$$R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A), \quad (2.2)$$

$$R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R). \quad (2.3)$$

If we denote by $R(b \otimes a) = a_R \otimes b_R$, for $a \in A, b \in B$, then (2.2) and (2.3) may be written as:

$$(aa')_R \otimes b_R = a_R a'_r \otimes (b_R)_r, \quad (2.4)$$

$$a_R \otimes (bb')_R = (a_R)_r \otimes b_r b'_R, \quad (2.5)$$

for all $a, a' \in A$ and $b, b' \in B$, where r is another copy of R . If we define a multiplication on $A \otimes B$, by $\mu_R = (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$, that is

$$(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b', \quad (2.6)$$

then this multiplication is associative and $1 \otimes 1$ is the unit. This algebra structure is denoted by $A \otimes_R B$ and is called the *twisted tensor product* of A and B . This construction works also if A and B are algebras in an arbitrary monoidal category.

If $A \otimes_{R_1} B, B \otimes_{R_2} C$ and $A \otimes_{R_3} C$ are twisted tensor products of algebras, the twisting maps R_1, R_2, R_3 are called *compatible* if they satisfy

$$(A \otimes_{R_2}) \circ (R_3 \otimes B) \circ (C \otimes_{R_1}) = (R_1 \otimes C) \circ (B \otimes_{R_3}) \circ (R_2 \otimes A),$$

see [19]. If this is the case, the maps

$$T_1: C \otimes (A \otimes_{R_1} B) \rightarrow (A \otimes_{R_1} B) \otimes C \quad \text{and} \quad T_2: (B \otimes_{R_2} C) \otimes A \rightarrow A \otimes (B \otimes_{R_2} C)$$

given by

$$T_1 := (A \otimes_{R_2}) \circ (R_3 \otimes B) \quad \text{and} \quad T_2 := (R_1 \otimes C) \circ (B \otimes_{R_3})$$

are also twisting maps and $A \otimes_{T_2} (B \otimes_{R_2} C) \equiv (A \otimes_{R_1} B) \otimes_{T_1} C$; this algebra is denoted by $A \otimes_{R_1} B \otimes_{R_2} C$. This construction may be iterated to an arbitrary number of factors, see [19] for complete detail.

We recall the following result from [10], to be used in the sequel:

Theorem 2.1. *Let A, B be two algebras. Then any twisting map $R: B \otimes A \rightarrow A \otimes B$ extends to a unique twisting map $\tilde{R}: \Omega B \otimes \Omega A \rightarrow \Omega A \otimes \Omega B$ which satisfies the conditions*

$$\tilde{R} \circ (d_B \otimes \Omega A) = (\varepsilon_A \otimes d_B) \circ \tilde{R}, \quad (2.7)$$

$$\tilde{R} \circ (\Omega B \otimes d_A) = (d_A \otimes \varepsilon_B) \circ \tilde{R}, \quad (2.8)$$

where d_A and d_B denote the differentials on the algebras of universal differential forms ΩA and ΩB , and $\varepsilon_A, \varepsilon_B$ stand for the gradings on ΩA and ΩB , respectively. Moreover, $\Omega A \otimes_{\tilde{R}} \Omega B$ is a graded differential algebra with differential $d(\varphi \otimes \omega) := d_A \varphi \otimes \omega + (-1)^{|\varphi|} \varphi \otimes d_B \omega$.

Finally, we recall the definition of the noncommutative $2n$ -planes introduced by Connes and Dubois-Violette in [11]. Consider $\theta \in \mathcal{M}_n(\mathbb{R})$ an antisymmetric matrix, $\theta = (\theta_{\mu\nu})$, $\theta_{\nu\mu} = -\theta_{\mu\nu}$, and let $C_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ be the associative algebra generated by $2n$ elements $\{z^{\mu}, \bar{z}^{\mu}\}_{\mu=1, \dots, n}$ with relations

$$\left. \begin{aligned} z^{\mu} z^{\nu} &= \lambda^{\mu\nu} z^{\nu} z^{\mu} \\ \bar{z}^{\mu} \bar{z}^{\nu} &= \lambda^{\mu\nu} \bar{z}^{\nu} \bar{z}^{\mu} \\ \bar{z}^{\mu} z^{\nu} &= \lambda^{\nu\mu} z^{\nu} \bar{z}^{\mu} \end{aligned} \right\} \quad \forall \mu, \nu = 1, \dots, n, \text{ being } \lambda^{\mu\nu} := e^{i\theta_{\mu\nu}}. \quad (2.9)$$

Note that $\lambda^{\nu\mu} = (\lambda^{\mu\nu})^{-1} = \overline{\lambda^{\mu\nu}}$ for $\mu \neq \nu$, and $\lambda^{\mu\mu} = 1$ by antisymmetry. The algebra $C_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ will be then referred to as the (algebra of complex polynomial functions on the) noncommutative $2n$ -plane \mathbb{R}_{θ}^{2n} . In fact, former relations define a deformation \mathbb{C}_{θ}^n of \mathbb{C}^n , so we can identify the noncommutative complex n -plane \mathbb{C}_{θ}^n with \mathbb{R}_{θ}^{2n} by writing

$$C_{\text{alg}}(\mathbb{C}_{\theta}^n) := C_{\text{alg}}(\mathbb{R}_{\theta}^{2n}).$$

As shown in [19], $C_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ may be written as an iterated twisted tensor product of n commutative (polynomial) algebras.

3. R -matrices and twistors

In the literature there exist various schemes producing, from a given associative algebra A and some datum corresponding to it, a new associative algebra structure on the vector space A . The aim of this section is to prove that there exists such a general scheme that produces the twisted tensor product starting from the ordinary tensor product. Our source of inspiration is the following result of Borchers from [5,6], which arose in his Hopf algebraic approach to vertex algebras:

Theorem 3.1. [5,6] *Let D be an algebra with multiplication denoted by $\mu_D = \mu$ and let $T: D \otimes D \rightarrow D \otimes D$ be a linear map satisfying the following conditions: $T(1 \otimes d) = 1 \otimes d$, $T(d \otimes 1) = d \otimes 1$, for all $d \in D$, and*

$$\mu_{23} \circ T_{12} \circ T_{13} = T \circ \mu_{23}: D \otimes D \otimes D \rightarrow D \otimes D, \quad (3.1)$$

$$\mu_{12} \circ T_{23} \circ T_{13} = T \circ \mu_{12}: D \otimes D \otimes D \rightarrow D \otimes D, \quad (3.2)$$

$$T_{12} \circ T_{13} \circ T_{23} = T_{23} \circ T_{13} \circ T_{12}: D \otimes D \otimes D \rightarrow D \otimes D \otimes D, \quad (3.3)$$

with standard notation for μ_{ij} and T_{ij} . Then the bilinear map $\mu \circ T: D \otimes D \rightarrow D$ is another associative algebra structure on D , with the same unit 1. The map T is called an R -matrix.

If $A \otimes_R B$ is a twisted tensor product of algebras, we want to obtain it as a twisting of $A \otimes B$. Define $T : (A \otimes B) \otimes (A \otimes B) \rightarrow (A \otimes B) \otimes (A \otimes B)$ by $T = (A \otimes \tau \otimes B) \circ (A \otimes R \otimes B)$, i.e.

$$T((a \otimes b) \otimes (a' \otimes b')) = (a \otimes b_R) \otimes (a'_R \otimes b'). \quad (3.4)$$

Then the multiplication of $A \otimes_R B$ is obtained as $\mu_{A \otimes B} \circ T$, also T satisfies $T(1 \otimes (a \otimes b)) = 1 \otimes (a \otimes b)$ and $T((a \otimes b) \otimes 1) = (a \otimes b) \otimes 1$, but in general T does not satisfy the other axioms in Theorem 3.1 (for instance take R to be the twisting map corresponding to a Hopf smash product), hence we cannot obtain $A \otimes_R B$ from $A \otimes B$ using Borchers' scheme, we have to find an alternative one. This is achieved in the next result (the proof is postponed to Section 6, where it will be given in a more general framework).

Theorem 3.2. *Let D be an algebra with multiplication denoted by $\mu_D = \mu$ and $T : D \otimes D \rightarrow D \otimes D$ a linear map satisfying the following conditions: $T(1 \otimes d) = 1 \otimes d$, $T(d \otimes 1) = d \otimes 1$, for all $d \in D$, and*

$$\mu_{23} \circ T_{13} \circ T_{12} = T \circ \mu_{23} : D \otimes D \otimes D \rightarrow D \otimes D, \quad (3.5)$$

$$\mu_{12} \circ T_{13} \circ T_{23} = T \circ \mu_{12} : D \otimes D \otimes D \rightarrow D \otimes D, \quad (3.6)$$

$$T_{12} \circ T_{23} = T_{23} \circ T_{12} : D \otimes D \otimes D \rightarrow D \otimes D \otimes D. \quad (3.7)$$

Then the bilinear map $\mu \circ T : D \otimes D \rightarrow D$ is another associative algebra structure on D , with the same unit 1, which will be denoted in what follows by D^T , and the map T will be called a twistor for D .

If T is a twistor, we will usually denote $T(d \otimes d') = d^T \otimes d'_T$, for $d, d' \in D$, so the new multiplication $\mu \circ T$ on D is given by $d * d' = d^T d'_T$. With this notation, the relations (3.5)–(3.7) may be written as:

$$d^T \otimes (d' d'')_T = (d^T)^t \otimes d'_T d''_t, \quad (3.8)$$

$$(d d')^T \otimes d''_T = d^T d'^t \otimes (d''_t)_T, \quad (3.9)$$

$$d^T \otimes (d'_T)^t \otimes d''_t = d^T \otimes (d'^t)_T \otimes d''_t. \quad (3.10)$$

Now, if $A \otimes_R B$ is a twisted tensor product of algebras, then one can check that the map T given by (3.4) satisfies the axioms in Theorem 3.2 for $D = A \otimes B$, and the deformed multiplication is the one of $A \otimes_R B$, that is $A \otimes_R B = (A \otimes B)^T$, so we obtained the associativity of $A \otimes_R B$ as a consequence of Theorem 3.2.

Conversely, if $R : B \otimes A \rightarrow A \otimes B$ is a linear map such that the map T given by (3.4) is a twistor for $A \otimes B$, then R is a twisting map and $(A \otimes B)^T = A \otimes_R B$. If this is the case, we will say that the twistor T is afforded by the twisting map R .

Remark 3.3. If T is a twistor for an algebra D , a consequence of (3.8) and (3.9) is:

$$T(ab \otimes cd) = (a^T)^t (b^T)^{\bar{T}} \otimes (c_T)_T (d_{\bar{T}})_t, \quad (3.11)$$

for all $a, b, c, d \in D$, where $T = t = T = \bar{T}$.

Remark 3.4. Let T be a twistor satisfying the extra conditions

$$T_{12} \circ T_{13} = T_{13} \circ T_{12}, \quad (3.12)$$

$$T_{13} \circ T_{23} = T_{23} \circ T_{13}. \quad (3.13)$$

Then it is easy to see that T is also an R -matrix. Conversely, a bijective R -matrix satisfying (3.12) and (3.13) is a twistor. An example of a twistor T satisfying (3.12) and (3.13) can easily be obtained as follows: take H a cocommutative bialgebra, $\sigma : H \otimes H \rightarrow k$ a bicharacter (i.e. σ satisfies $\sigma(1, h) = \sigma(h, 1) = \varepsilon(h)$, $\sigma(h, h'h'') = \sigma(h_1, h')\sigma(h_2, h'')$ and $\sigma(hh', h'') = \sigma(h, h'_1)\sigma(h', h''_2)$ for all $h, h', h'' \in H$) and $T : H \otimes H \rightarrow H \otimes H$, $T(h \otimes h') = \sigma(h_1, h'_1)h_2 \otimes h'_2$.

Remark 3.5. We have seen before (formula (3.4)) a basic example of a twistor which in general is not an R -matrix. We present now a basic example of an R -matrix which is not a twistor. Namely, for any algebra D , define the map $T : D \otimes D \rightarrow D \otimes D$, $T(d \otimes d') = d'd \otimes 1 + 1 \otimes d'd - d' \otimes d$. Then one can check that T is an R -matrix (the fact that it satisfies (3.3) follows from [26] or [25]) and is not a twistor. Note that the multiplication $\mu \circ T$ afforded by T is just the multiplication of the opposite algebra D^{op} .

4. More examples of twistors

In this section we present more situations where Theorem 3.2 may be applied.

(i) Let A, B, C be three algebras and $R_1 : B \otimes A \rightarrow A \otimes B$, $R_2 : C \otimes B \rightarrow B \otimes C$, $R_3 : C \otimes A \rightarrow A \otimes C$ twisting maps. Consider the algebra $D = A \otimes B \otimes C$ and the map $T : D \otimes D \rightarrow D \otimes D$,

$$T((a \otimes b \otimes c) \otimes (a' \otimes b' \otimes c')) = (a \otimes b_{R_1} \otimes (c_{R_3})_{R_2}) \otimes ((a'_{R_3})_{R_1} \otimes b'_{R_2} \otimes c'). \quad (4.1)$$

In general T is not a twistor for D , even if the maps R_1, R_2, R_3 are compatible. But we have the following result:

Proposition 4.1. *With notation as above, T is a twistor for D if and only if the following conditions hold:*

$$a_{R_1} \otimes (b_{R_1})_{R_2} \otimes c_{R_2} = a_{R_1} \otimes (b_{R_2})_{R_1} \otimes c_{R_2}, \quad (4.2)$$

$$(a_{R_1})_{R_3} \otimes b_{R_1} \otimes c_{R_3} = (a_{R_3})_{R_1} \otimes b_{R_1} \otimes c_{R_3}, \quad (4.3)$$

$$a_{R_3} \otimes b_{R_2} \otimes (c_{R_3})_{R_2} = a_{R_3} \otimes b_{R_2} \otimes (c_{R_2})_{R_3}, \quad (4.4)$$

for all $a \in A$, $b \in B$, $c \in C$. Moreover, in this case it follows that R_1, R_2, R_3 are compatible twisting maps and $D^T = A \otimes_{R_1} B \otimes_{R_2} C$.

Proof. The fact that T is a twistor if and only if (4.2)–(4.4) hold follows by a direct computation, we leave the details to the reader. We only prove that R_1, R_2, R_3 are compatible. We compute:

$$\begin{aligned} (A \otimes R_2)(R_3 \otimes B)(C \otimes R_1)(a \otimes b \otimes c) &= (a_{R_1})_{R_3} \otimes (b_{R_1})_{R_2} \otimes (c_{R_3})_{R_2} \\ &\stackrel{(4.2)}{=} (a_{R_1})_{R_3} \otimes (b_{R_2})_{R_1} \otimes (c_{R_3})_{R_2} \end{aligned}$$

$$\begin{aligned}
&\stackrel{(4.3)}{=} (a_{R_3})_{R_1} \otimes (b_{R_2})_{R_1} \otimes (c_{R_3})_{R_2} \\
&\stackrel{(4.4)}{=} (a_{R_3})_{R_1} \otimes (b_{R_2})_{R_1} \otimes (c_{R_2})_{R_3} \\
&= (R_1 \otimes C)(B \otimes R_3)(R_2 \otimes A)(a \otimes b \otimes c).
\end{aligned}$$

The fact that $D^T = A \otimes_{R_1} B \otimes_{R_2} C$ is obvious. \square

Remark 4.2. The conditions in Proposition 4.1 are satisfied whenever we start with compatible twisting maps R_1, R_2, R_3 such that one of them is a usual flip; a concrete example where this happens is for the so-called two-sided smash product, see [19] for details.

Proposition 4.1 may be extended to an iterated twisted tensor product of any number of factors by means of the Coherence Theorem stated in [19]. In order to do this, just realize that conditions (4.2), (4.3), and (4.4) mean simply requiring that $\{R_1, R_2, \tau_{AC}\}$, $\{R_1, \tau_{BC}, R_3\}$ and $\{\tau_{AB}, R_2, R_3\}$ are sets of compatible twisting maps, where the τ 's are classical flips.

Proposition 4.3. Let A_1, \dots, A_n be some algebras, $\{R_{ij}\}_{i < j}$ a set of twisting maps, with $R_{ij}: A_j \otimes A_i \rightarrow A_i \otimes A_j$, and let $D = A_1 \otimes \dots \otimes A_n$. Then the following two conditions are equivalent:

(1) The map $T: D \otimes D \rightarrow D \otimes D$ defined by

$$\begin{aligned}
T := & (Id_{A_1 \otimes \dots \otimes A_{n-1}} \otimes \tau_{n1} \otimes Id_{A_2 \otimes \dots \otimes A_n}) \circ \dots \\
& \circ (Id_{A_1 \otimes \dots \otimes A_{n-k-1}} \otimes \tau_{n-k1} \otimes \dots \otimes \tau_{nk+1} \otimes Id_{A_{k+2} \otimes \dots \otimes A_n}) \circ \dots \\
& \circ (Id_{A_1} \otimes \tau_{21} \otimes \dots \otimes \tau_{nn-1} \otimes Id_{A_n}) \circ (Id_{A_1} \otimes R_{12} \otimes \dots \otimes R_{n-1n} \otimes Id_{A_n}) \circ \dots \\
& \circ (Id_{A_1 \otimes \dots \otimes A_{n-k-1}} \otimes R_{1n-k} \otimes \dots \otimes R_{k+1} \otimes Id_{A_{k+2} \otimes \dots \otimes A_n}) \circ \dots \\
& \circ (Id_{A_1 \otimes \dots \otimes A_{n-1}} \otimes R_{n1} \otimes Id_{A_2 \otimes \dots \otimes A_n})
\end{aligned}$$

is a twistor.

(2) For any triple $i < j < k \in \{1, \dots, n\}$, we have that $\{R_{ij}, R_{jk}, \tau_{ik}\}$, $\{R_{ij}, \tau_{jk}, R_{ik}\}$ and $\{\tau_{ij}, R_{jk}, R_{ik}\}$ are sets of compatible twisting maps.

Moreover, if the conditions are satisfied, then the twisting maps $\{R_{ij}\}_{i < j}$ are compatible, and we have $D^T = A_1 \otimes_{R_{12}} \dots \otimes_{R_{n-1n}} A_n$, that is, the twisting induced by the twistor T gives the iterated twisted tensor product associated to the maps.

Proof. We just outline the main ideas of the proof, leaving details to the reader. The proof is by induction on the number of terms $n \geq 3$; for $n = 3$, the result is just Proposition 4.1. Now, assuming the result is true for $n - 1$ algebras with their corresponding twisting maps, and given A_1, \dots, A_n algebras, satisfying the hypothesis of the proposition, we consider the algebras $B_1 := A_1, \dots, B_{n-2} := A_{n-2}, B_{n-1} := A_{n-1} \otimes_{R_{n-1n}} A_n$, with the twisting maps defined as in the Coherence Theorem. Directly from the hypothesis of the proposition, it follows from the Coherence Theorem that the newly defined twisting maps also satisfy the conditions in the proposition, so we may apply our induction hypothesis to the algebras B_1, \dots, B_{n-1} . \square

A particular case of the former proposition is found in the realization of the noncommutative planes of Connes and Dubois-Violette as iterated twisted tensor products [19]. As the twisting maps involved in this process are just multiples of the classical flips, the compatibility conditions are trivially satisfied, and the proposition tells us that any noncommutative $2n$ -plane $C_{\text{alg}}(\mathbb{R}_\theta^{2n})$ may also be realized as a deformation through a twistor of the commutative algebra $\mathbb{C}[z^1, \bar{z}^1, \dots, z^n, \bar{z}^n]$. Moreover, the former proposition provides an explicit formula for the twistor T that recovers the iterated twisted tensor product. Taking into account the identification

$$\begin{aligned}\mathbb{C}[z^1, \bar{z}^1, \dots, z^n, \bar{z}^n] &\rightarrow \mathbb{C}[z^1, \bar{z}^1] \otimes \cdots \otimes \mathbb{C}[z^n, \bar{z}^n], \\ z^i &\mapsto 1 \otimes \cdots \otimes z^i \otimes \cdots \otimes 1, \\ \bar{z}^i &\mapsto 1 \otimes \cdots \otimes \bar{z}^i \otimes \cdots \otimes 1,\end{aligned}$$

where z^i maps to the position $2i - 1$ and \bar{z}^i maps to the position $2i$, it is easy to realize that the twistor given by the proposition is defined on generators as:

$$\begin{aligned}T(z^i \otimes z^j) &= \begin{cases} z^i \otimes z^j & \text{if } i \leq j, \\ \lambda^{ij} z^i \otimes z^j & \text{otherwise,} \end{cases} & T(\bar{z}^i \otimes \bar{z}^j) &= \begin{cases} \bar{z}^i \otimes \bar{z}^j & \text{if } i \leq j, \\ \lambda^{ij} \bar{z}^i \otimes \bar{z}^j & \text{otherwise,} \end{cases} \\ T(\bar{z}^i \otimes z^j) &= \begin{cases} \bar{z}^i \otimes z^j & \text{if } i \leq j, \\ \lambda^{ji} \bar{z}^i \otimes z^j & \text{otherwise,} \end{cases} & T(z^i \otimes \bar{z}^j) &= \begin{cases} z^i \otimes \bar{z}^j & \text{if } i \leq j, \\ \lambda^{ji} z^i \otimes \bar{z}^j & \text{otherwise.} \end{cases}\end{aligned}$$

(ii) Let A be an algebra with multiplication $\mu_A = \mu$ and H a bialgebra such that A is an H -bimodule algebra with actions denoted by $\pi_l: H \otimes A \rightarrow A$, $\pi_l(h \otimes a) = h \cdot a$ and $\pi_r: A \otimes H \rightarrow A$, $\pi_r(a \otimes h) = a \cdot h$, also A is an H -bicomodule algebra, with coactions denoted by $\psi_l: A \rightarrow H \otimes A$, $a \mapsto a_{[-1]} \otimes a_{[0]}$ and $\psi_r: A \rightarrow A \otimes H$, $a \mapsto a_{(0)} \otimes a_{(1)}$, and moreover the following compatibility conditions hold, for all $h \in H$ and $a \in A$:

$$\begin{aligned}(h \cdot a)_{[-1]} \otimes (h \cdot a)_{[0]} &= a_{[-1]} \otimes h \cdot a_{[0]}, & (h \cdot a)_{(0)} \otimes (h \cdot a)_{(1)} &= h \cdot a_{(0)} \otimes a_{(1)}, \\ (a \cdot h)_{[-1]} \otimes (a \cdot h)_{[0]} &= a_{[-1]} \otimes a_{[0]} \cdot h, & (a \cdot h)_{(0)} \otimes (a \cdot h)_{(1)} &= a_{(0)} \cdot h \otimes a_{(1)}.\end{aligned}$$

Such a datum was considered in [27], where it is called an L-R-twisting datum for A (and contains as particular case the concept of very strong left twisting datum from [16], which is obtained if the right action and coaction are trivial).

Proposition 4.4. [27] *Given an L-R-twisting datum, define a new multiplication on A by*

$$a \bullet a' = (a_{[0]} \cdot a'_{(1)})(a_{[-1]} \cdot a'_{(0)}), \quad \forall a, a' \in A. \quad (4.5)$$

Then $(A, \bullet, 1)$ is an associative unital algebra.

This result may be obtained as a consequence of Theorem 3.2. Namely, define

$$T: A \otimes A \rightarrow A \otimes A, \quad T(a \otimes a') = a_{[0]} \cdot a'_{(1)} \otimes a_{[-1]} \cdot a'_{(0)}. \quad (4.6)$$

Then one can check that T is a twistor for A , and obviously the new multiplication \bullet defined above coincides with $\mu \circ T$.

(iii) Let H, K be two bialgebras, A an algebra which is a left H -comodule algebra with coaction $a \mapsto a_{[-1]} \otimes a_{[0]} \in H \otimes A$ and a left K -module algebra with action $k \otimes a \mapsto k \cdot a$, for all $a \in A, k \in K$, such that $(k \cdot a)_{[-1]} \otimes (k \cdot a)_{[0]} = a_{[-1]} \otimes k \cdot a_{[0]}$, for all $a \in A, k \in K$. Let $f: H \rightarrow K$ be a bialgebra map. Then, by [9], the new multiplication defined on A by $a \cdot_f a' = a_{[0]}(f(a_{[-1]}) \cdot a')$ is associative with unit 1. This multiplication is afforded by the map $T: A \otimes A \rightarrow A \otimes A, T(a \otimes a') = a_{[0]} \otimes f(a_{[-1]}) \cdot a'$, which is easily seen to be a twistor.

(iv) Let H be a bialgebra and $F = F^1 \otimes F^2 \in H \otimes H$ an element with $(\varepsilon \otimes H)(F) = (H \otimes \varepsilon)(F) = 1$. Assume that F satisfies the following list of axioms, considered in [18,22]: $(H \otimes \Delta)(F) = F_{13}F_{12}, (\Delta \otimes H)(F) = F_{13}F_{23}$ and $F_{12}F_{23} = F_{23}F_{12}$. Let D be a left H -module algebra and define $T: D \otimes D \rightarrow D \otimes D$ by $T(d \otimes d') = F^1 \cdot d \otimes F^2 \cdot d'$. Then it is easy to see that T is a twistor for D . In case F is invertible, the multiplication of D^T fits into the well-known procedure of twisting a module algebra by a Drinfeld twist.

(v) Let H be a bialgebra and $\sigma: H \otimes H \rightarrow k$ a linear map. Define $T: H \otimes H \rightarrow H \otimes H$ by $T(a \otimes b) = \sigma(a_1, b_1)a_2 \otimes b_2$, for all $a, b \in H$. Then T is a twistor for H if and only if σ satisfies the following conditions: $\sigma(a, 1) = \varepsilon(a) = \sigma(1, a), \sigma(a, bc) = \sigma(a_1, b)\sigma(a_2, c), \sigma(ab, c) = \sigma(a, c_2)\sigma(b, c_1)$ and $\sigma(a, b_1)\sigma(b_2, c) = \sigma(b_1, c)\sigma(a, b_2)$, for all $a, b, c \in H$. Note that elements satisfying the last condition have been considered in [28], under the name *neat* elements.

(vi) Let (D, δ) be a differential associative algebra, that is D is an associative algebra and $\delta: D \rightarrow D$ is a derivation (i.e. $\delta(dd') = \delta(d)d' + d\delta(d')$) with $\delta^2 = 0$. Then one can see that the map $T: D \otimes D \rightarrow D \otimes D, T(d \otimes d') = d \otimes d' + \delta(d) \otimes \delta(d')$ is a twistor for D .

5. Some properties of twistors

Proposition 5.1. *Let T be a twistor for an algebra D and U a twistor for an algebra F . If $v: D \rightarrow F$ is an algebra map such that $(v \otimes v) \circ T = U \circ (v \otimes v)$, then v is also an algebra map from D^T to F^U .*

It was proved in [7] that, if $A \otimes_R B$ and $A' \otimes_{R'} B'$ are twisted tensor products of algebras and $f: A \rightarrow A'$ and $g: B \rightarrow B'$ are algebra maps satisfying the condition $(f \otimes g) \circ R = R' \circ (g \otimes f)$, then $f \otimes g: A \otimes_R B \rightarrow A' \otimes_{R'} B'$ is an algebra map. One can easily see that this result is a particular case of Proposition 5.1, with $D = A \otimes B, F = A' \otimes B', v = f \otimes g$ and T (respectively U) the twistor afforded by R (respectively R').

We present one more situation where Proposition 5.1 may be applied. We recall that the L-R-smash product over a cocommutative Hopf algebra was introduced in [3,4], and generalized to an arbitrary Hopf algebra in [27] as follows: if A is an H -bimodule algebra, the L-R-smash product $A \bowtie H$ is the following algebra structure on $A \otimes H$:

$$(a \bowtie h)(a' \bowtie h') = (a \cdot h'_2)(h_1 \cdot a') \bowtie h_2 h'_1, \quad \forall a, a' \in A, h, h' \in H.$$

The diagonal crossed product $A \bowtie H$ is the following algebra structure on $A \otimes H$, see [8,17]:

$$(a \bowtie h)(a' \bowtie h') = a(h_1 \cdot a' \cdot S^{-1}(h_3)) \bowtie h_2 h'_1, \quad \forall a, a' \in A, h, h' \in H.$$

It was proved in [27] that actually $A \bowtie H$ and $A \bowtie H$ are isomorphic as algebras. This result may be reobtained using Proposition 5.1 as follows. Denote by $A \#_r H$ the algebra structure on $A \otimes H$ with multiplication $(a \otimes h)(a' \otimes h') = (a \cdot h'_2)a' \otimes h h'_1$, and by $A \bowtie_r H$ the algebra

structure on $A \otimes H$ with multiplication $(a \otimes h)(a' \otimes h') = a(a' \cdot S^{-1}(h_2)) \otimes h_1 h'$. One may check that the map $\nu: A \bowtie_r H \rightarrow A \#_r H$ given by $\nu(a \otimes h) = a \cdot h_2 \otimes h_1$ is an algebra map (actually, an isomorphism, with inverse $\nu^{-1}(a \otimes h) = a \cdot S^{-1}(h_2) \otimes h_1$). Define now the map $T: (A \otimes H) \otimes (A \otimes H) \rightarrow (A \otimes H) \otimes (A \otimes H)$ by $T((a \otimes h) \otimes (a' \otimes h')) = (a \otimes h_2) \otimes (h_1 \cdot a' \otimes h')$. Then one may check, by direct computation, that T is a twistor for both $A \#_r H$ and $A \bowtie_r H$, and moreover $(A \#_r H)^T = A \natural H$, $(A \bowtie_r H)^T = A \bowtie H$ and $(\nu \otimes \nu) \circ T = T \circ (\nu \otimes \nu)$. Hence, Proposition 5.1 may be applied and we obtain as a consequence that ν is an algebra map from $A \bowtie H$ to $A \natural H$.

By [27], the L-R-twisted product (4.5) may be obtained as a left twisting followed by a right twisting and vice versa. This fact admits an interpretation in terms of twistors.

Proposition 5.2. *Let D be an algebra and $X, Y: D \otimes D \rightarrow D \otimes D$ two twistors for D , satisfying the following conditions:*

$$X_{23} \circ Y_{12} = Y_{12} \circ X_{23}, \quad (5.1)$$

$$X_{23} \circ Y_{13} = Y_{13} \circ X_{23}, \quad (5.2)$$

$$X_{12} \circ Y_{23} = Y_{23} \circ X_{12}, \quad (5.3)$$

$$X_{12} \circ Y_{13} = Y_{13} \circ X_{12}. \quad (5.4)$$

Then Y is a twistor for D^X , X is a twistor for D^Y , $X \circ Y$ and $Y \circ X$ are twistors for D and of course $(D^X)^Y = D^{X \circ Y}$ and $(D^Y)^X = D^{Y \circ X}$.

Proof. Note first that (5.2) and (5.4) are respectively equivalent to $X_{13} \circ Y_{23} = Y_{23} \circ X_{13}$ and $Y_{12} \circ X_{13} = X_{13} \circ Y_{12}$, hence the above conditions are actually symmetric in X and Y , so we only have to prove that Y is a twistor for D^X and $X \circ Y$ is a twistor for D .

To prove that Y is a twistor for D^X we only have to check (3.8) and (3.9) for Y with respect to the multiplication $*$ of D^X ; we compute:

$$\begin{aligned} d^Y \otimes (d' * d'')_Y &= d^Y \otimes (d'^X d''_X)_Y \\ &\stackrel{(3.8)}{=} (d^Y)^Y \otimes (d'^X)_Y (d''_X)_Y \\ &\stackrel{(5.2)}{=} (d^Y)^Y \otimes (d'^X)_Y (d''_Y)_X \\ &\stackrel{(5.1)}{=} (d^Y)^Y \otimes (d'_Y)^X (d''_Y)_X \\ &= (d^Y)^Y \otimes d'_Y * d''_Y, \\ (d * d')^Y \otimes d''_Y &= (d^X d'_X)^Y \otimes d''_Y \\ &\stackrel{(3.9)}{=} (d^X)^Y (d'_X)^Y \otimes (d''_Y)_Y \\ &\stackrel{(5.3)}{=} (d^X)^Y (d'^Y)_X \otimes (d''_Y)_Y \\ &\stackrel{(5.4)}{=} (d^Y)^X (d'^Y)_X \otimes (d''_Y)_Y \\ &= d^Y * d'^Y \otimes (d''_Y)_Y. \end{aligned}$$

Now we check (3.8) and (3.9) for $T := X \circ Y$; we compute:

$$\begin{aligned}
 d^T \otimes (d' d'')_T &= (d^Y)^X \otimes ((d' d'')_Y)_X \\
 &\stackrel{(3.8)}{=} ((d^Y)^Y)^X \otimes (d'_Y d''_Y)_X \\
 &\stackrel{(3.8)}{=} (((d^Y)^Y)^X)^X \otimes (d'_Y)_X (d''_Y)_X \\
 &\stackrel{(5.4)}{=} (((d^Y)^X)^Y)^X \otimes (d'_Y)_X (d''_Y)_X \\
 &= (d^T)^t \otimes d'_T d''_T, \\
 (dd')^T \otimes d''_T &= ((dd')^Y)^X \otimes (d''_Y)_X \\
 &\stackrel{(3.9)}{=} (d^Y d'^Y)^X \otimes ((d''_Y)_Y)_X \\
 &\stackrel{(3.9)}{=} (d^Y)^X (d'^Y)^X \otimes (((d''_Y)_Y)_X)_X \\
 &\stackrel{(5.2)}{=} (d^Y)^X (d'^Y)^X \otimes (((d''_Y)_X)_Y)_X \\
 &= d^T d'^t \otimes (d''_t)_T.
 \end{aligned}$$

It remains to prove (3.7) for T ; we compute:

$$\begin{aligned}
 T_{12} \circ T_{23} &= X_{12} \circ Y_{12} \circ X_{23} \circ Y_{23} \\
 &\stackrel{(5.1)}{=} X_{12} \circ X_{23} \circ Y_{12} \circ Y_{23} \\
 &\stackrel{(3.7)}{=} X_{23} \circ X_{12} \circ Y_{23} \circ Y_{12} \\
 &\stackrel{(5.3)}{=} X_{23} \circ Y_{23} \circ X_{12} \circ Y_{12} \\
 &= T_{23} \circ T_{12},
 \end{aligned}$$

and the proof is finished. \square

Let now A be as in Proposition 4.4 and define $X, Y : A \otimes A \rightarrow A \otimes A$ by

$$X(a \otimes a') = a \cdot a'_{(1)} \otimes a'_{(0)}, \quad Y(a \otimes a') = a_{[0]} \otimes a_{[-1]} \cdot a'.$$

Then one can check that X and Y satisfy the hypotheses of Proposition 5.2, and moreover we have $X \circ Y = Y \circ X = T$, where T is given by (4.6). Hence, we obtain

$$(A, \bullet, 1) = (A^X)^Y = (A^Y)^X.$$

Also as a consequence of Proposition 5.2, we obtain that if T is a twistor for an algebra D , satisfying (3.12) and (3.13), then T is a twistor also for D^T , hence we obtain a sequence of associative algebras D, D^T, D^{T^2}, D^{T^3} , etc.

A particular case of Proposition 5.2 is the following:

Corollary 5.3. *Let A, B be two algebras and $R, S: B \otimes A \rightarrow A \otimes B$ two twisting maps. Denote by X (respectively Y) the twistor for $A \otimes B$ afforded by R (respectively S) and assume that the following conditions are satisfied:*

$$(a_R)_S \otimes b_R \otimes b'_S = (a_S)_R \otimes b_R \otimes b'_S, \quad a_R \otimes a'_S \otimes (b_R)_S = a_R \otimes a'_S \otimes (b_S)_R,$$

for all $a, a' \in A$ and $b, b' \in B$. Define $R * S, S * R: B \otimes A \rightarrow A \otimes B$ by

$$(R * S)(b \otimes a) = (a_S)_R \otimes (b_S)_R, \quad (S * R)(b \otimes a) = (a_R)_S \otimes (b_R)_S.$$

Then Y is a twistor for $A \otimes_R B$, X is a twistor for $A \otimes_S B$, $X \circ Y$ (respectively $Y \circ X$) is a twistor for $A \otimes B$ afforded by the twisting map $R * S$ (respectively $S * R$) and we have $(A \otimes_R B)^Y = A \otimes_{R*S} B$, $(A \otimes_S B)^X = A \otimes_{S*R} B$.

We are now interested in lifting (bi)module structures from an algebra D to D^T . This is achieved in the next result, the proof follows from a direct computation and will be omitted.

Proposition 5.4. *Let D be an algebra and T a twistor for D .*

(i) *Let V be a left D -module, with action $\lambda: D \otimes V \rightarrow V$, $\lambda(d \otimes v) = d \cdot v$. Assume that we are given a linear map $\Gamma: D \otimes V \rightarrow D \otimes V$, with notation $\Gamma(d \otimes v) = d_\Gamma \otimes v_\Gamma$, for all $d \in D$, $v \in V$, such that $\Gamma(1 \otimes v) = 1 \otimes v$, for all $v \in V$, and*

$$\lambda_{23} \circ \Gamma_{13} \circ T_{12} = \Gamma \circ \lambda_{23}: D \otimes D \otimes V \rightarrow D \otimes V, \quad (5.5)$$

$$\mu_{12} \circ \Gamma_{13} \circ \Gamma_{23} = \Gamma \circ \mu_{12}: D \otimes D \otimes V \rightarrow D \otimes V, \quad (5.6)$$

$$T_{12} \circ \Gamma_{23} = \Gamma_{23} \circ T_{12}: D \otimes D \otimes V \rightarrow D \otimes D \otimes V. \quad (5.7)$$

Then V becomes a left D^T -module, with action $\lambda \circ \Gamma: D \otimes V \rightarrow V$. We denote by V^Γ this D^T -module structure on V and by $d \rightarrow v = d_\Gamma \cdot v_\Gamma$ the action of D^T on V . We call the map Γ a left module twistor for V relative to T .

(ii) *Let V be a right D -module, with action $\rho: V \otimes D \rightarrow V$, $\rho(v \otimes d) = v \cdot d$, and assume that we are given a linear map $\Pi: V \otimes D \rightarrow V \otimes D$, with notation $\Pi(v \otimes d) = v_\Pi \otimes d_\Pi$, for all $d \in D$, $v \in V$, such that $\Pi(v \otimes 1) = v \otimes 1$, for all $v \in V$, and*

$$\mu_{23} \circ \Pi_{13} \circ \Pi_{12} = \Pi \circ \mu_{23}: V \otimes D \otimes D \rightarrow V \otimes D, \quad (5.8)$$

$$\rho_{12} \circ \Pi_{13} \circ T_{23} = \Pi \circ \rho_{12}: V \otimes D \otimes D \rightarrow V \otimes D, \quad (5.9)$$

$$\Pi_{12} \circ T_{23} = T_{23} \circ \Pi_{12}: V \otimes D \otimes D \rightarrow V \otimes D \otimes D. \quad (5.10)$$

Then V becomes a right D^T -module, with action $\rho \circ \Pi: V \otimes D \rightarrow V$. We denote by ${}^\Pi V$ this D^T -module structure on V and by $v \leftarrow d = v_\Pi \cdot d_\Pi$ the action of D^T on V . We call the map Π a right module twistor for V relative to T .

(iii) *Let V be a D -bimodule, and let Γ and Π be a left respectively a right module twistor for V relative to T . Assume that the following conditions hold:*

$$\rho_{23} \circ T_{13} \circ \Gamma_{12} = \Gamma \circ \rho_{23} : D \otimes V \otimes D \rightarrow D \otimes V, \quad (5.11)$$

$$\lambda_{12} \circ T_{13} \circ \Pi_{23} = \Pi \circ \lambda_{12} : D \otimes V \otimes D \rightarrow V \otimes D, \quad (5.12)$$

$$\Gamma_{12} \circ \Pi_{23} = \Pi_{23} \circ \Gamma_{12} : D \otimes V \otimes D \rightarrow D \otimes V \otimes D. \quad (5.13)$$

Let ${}^{\Pi}V^{\Gamma}$ be V^{Γ} as a left D^T -module and ${}^{\Pi}V$ as a right D^T -module. Then ${}^{\Pi}V^{\Gamma}$ is a D^T -bimodule.

We recall from [10] the following result. Let $A \otimes_R B$ be a twisted tensor product of algebras, M a left A -module, N a left B -module (we denote by λ_M and respectively λ_N the actions) and $\tau_{M,B} : B \otimes M \rightarrow M \otimes B$ a linear map, with notation $\tau_{M,B}(b \otimes m) = m_{\tau} \otimes b_{\tau}$, such that $\tau_{M,B}(1 \otimes m) = m \otimes 1$, for all $m \in M$, and the following conditions hold:

$$\tau_{M,B} \circ (\mu_B \otimes M) = (M \otimes \mu_B) \circ (\tau_{M,B} \otimes B) \circ (B \otimes \tau_{M,B}),$$

$$\tau_{M,B} \circ (B \otimes \lambda_M) = (\lambda_M \otimes B) \circ (A \otimes \tau_{M,B}) \circ (R \otimes M)$$

(such a map $\tau_{M,B}$ is called a left module twisting map). Then $M \otimes N$ becomes a left $A \otimes_R B$ -module, with action $(a \otimes b) \cdot (m \otimes n) = a \cdot m_{\tau} \otimes b_{\tau} \cdot n$. This result is a particular case of Proposition 5.4(i). Indeed, we consider the algebra $D = A \otimes B$ (the ordinary tensor product), the twistor T for D given by (3.4), the left D -module $V = M \otimes N$ with action $(a \otimes b) \cdot (m \otimes n) = a \cdot m \otimes b \cdot n$, and the map $\Gamma : (A \otimes B) \otimes (M \otimes N) \rightarrow (A \otimes B) \otimes (M \otimes N)$ given by $\Gamma((a \otimes b) \otimes (m \otimes n)) = (a \otimes b_{\tau}) \otimes (m_{\tau} \otimes n)$. Then one can check that Γ satisfies the axioms of a left module twistor, and the left $D^T = A \otimes_R B$ -module V^{Γ} is obviously the $A \otimes_R B$ -module structure on $M \otimes N$ presented above. Similarly, one can see that Proposition 5.4(ii) contains as particular case the lifting of right module structures to a twisted tensor product from [10].

Another example may be obtained as follows. Let A be as in Proposition 4.4, and V a vector space which is a left A -module (with action $a \otimes v \mapsto a \cdot v$), a left H -module (with action $h \otimes v \mapsto h \cdot v$) and a right H -comodule (with coaction $v \mapsto v_{(0)} \otimes v_{(1)} \in V \otimes H$) such that the following conditions are satisfied, for all $h \in H$, $a \in A$, $v \in V$:

$$(h \cdot v)_{(0)} \otimes (h \cdot v)_{(1)} = h \cdot v_{(0)} \otimes v_{(1)},$$

$$h \cdot (a \cdot v) = (h_1 \cdot a) \cdot (h_2 \cdot v),$$

$$(a \cdot v)_{(0)} \otimes (a \cdot v)_{(1)} = a_{(0)} \cdot v_{(0)} \otimes a_{(1)} v_{(1)}.$$

Define the map $\Gamma : A \otimes V \rightarrow A \otimes V$ by $\Gamma(a \otimes v) = a_{[0]} \cdot v_{(1)} \otimes a_{[-1]} \cdot v_{(0)}$. Then one can check that Γ and the twistor T given by (4.6) satisfy the hypotheses of Proposition 5.4(i), hence V becomes a left module over (A, \bullet) , with action $a \rightarrow v = (a_{[0]} \cdot v_{(1)}) \cdot (a_{[-1]} \cdot v_{(0)})$.

We present now an application of Proposition 5.4.

Proposition 5.5. *Let (D, μ, u) be an algebra and consider the universal first order differential calculus $\Omega_u^1(D) = \text{Ker}(\mu)$, with its canonical D -bimodule structure. If T is a twistor for D , then $\Omega_u^1(D)$ becomes also a D^T -bimodule.*

Proof. Define the maps $\Gamma, \Pi : D \otimes D \otimes D \rightarrow D \otimes D \otimes D$ by $\Gamma = T_{13} \circ T_{12}$ and $\Pi = T_{13} \circ T_{23}$. We claim that $\Gamma(D \otimes \text{Ker}(\mu)) \subseteq D \otimes \text{Ker}(\mu)$ and $\Pi(\text{Ker}(\mu) \otimes D) \subseteq \text{Ker}(\mu) \otimes D$. To prove this,

we recall the following result from linear algebra: if $f: V \rightarrow V'$ and $g: W \rightarrow W'$ are linear maps, then $\text{Ker}(f \otimes g) = \text{Ker}(f) \otimes W + V \otimes \text{Ker}(g)$. We apply this result for the map $D \otimes \mu: D \otimes D \otimes D \rightarrow D \otimes D \otimes D$, and we obtain $\text{Ker}(D \otimes \mu) = \text{Ker}(D) \otimes D \otimes D + D \otimes \text{Ker}(\mu) = D \otimes \text{Ker}(\mu)$. Let $x \in D \otimes \text{Ker}(\mu)$; in order to prove that $\Gamma(x) \in D \otimes \text{Ker}(\mu)$, in view of the above it is enough to prove that $((D \otimes \mu) \circ \Gamma)(x) = 0$. But using (3.5) and the definition of Γ , we see that $(D \otimes \mu) \circ \Gamma = T \circ \mu_{23}$, and obviously $(T \circ \mu_{23})(x) = 0$ because $x \in D \otimes \text{Ker}(\mu)$. Similarly one can prove that $\Pi(\text{Ker}(\mu) \otimes D) \subseteq \text{Ker}(\mu) \otimes D$. Now, if we denote by $\lambda: D \otimes \text{Ker}(\mu) \rightarrow \text{Ker}(\mu)$ and $\rho: \text{Ker}(\mu) \otimes D \rightarrow \text{Ker}(\mu)$ the left and right D -module structures of $\text{Ker}(\mu)$ (given by $\lambda = \mu_{12}$ and $\rho = \mu_{23}$), then the maps $\lambda, \rho, \Gamma, \Pi$ satisfy all the hypotheses of Proposition 5.4 (this proof is a direct computation and is omitted), hence indeed $\text{Ker}(\mu)$ becomes a D^T -bimodule. \square

Actually, more can be said about this D^T -bimodule $\text{Ker}(\mu)$. Denote by $\delta: D \rightarrow \text{Ker}(\mu)$, $\delta(d) = d \otimes 1 - 1 \otimes d$ the canonical D -derivation.

Proposition 5.6. *This map δ is also a D^T -derivation from D^T to $\text{Ker}(\mu)$, where the D^T -bimodule structure on $\text{Ker}(\mu)$ is the one presented above.*

Proof. Using the formulae for Γ and Π , one can easily see that $d \rightarrow \delta(d') = d^T \cdot \delta(d'_T)$ and $\delta(d) \leftarrow d' = \delta(d^T) \cdot d'_T$ for all $d, d' \in D$, so we immediately obtain:

$$\begin{aligned} \delta(d * d') &= \delta(d^T d'_T) \\ &= d^T \cdot \delta(d'_T) + \delta(d^T) \cdot d'_T \\ &= d \rightarrow \delta(d') + \delta(d) \leftarrow d', \end{aligned}$$

finishing the proof. \square

Proposition 5.7. *If the twistor T is bijective, then $(\text{Ker}(\mu), \delta)$ is also a first order differential calculus over the algebra D^T .*

Proof. We only have to prove that $\text{Ker}(\mu)$ is generated by $\{\delta(d): d \in D\}$ as a D^T -bimodule. If $d, d' \in D$, we denote by $T^{-1}(d \otimes d') = d^U \otimes d'_U$. If $x = \sum_i a_i \otimes b_i \in \text{Ker}(\mu)$, we can write $x = \sum_i \delta(a_i) \cdot b_i$, which in turn may be written as $x = \sum_i \delta(a_i^U) \leftarrow (b_i)_U$. \square

6. Pseudotwistors and braided (graded) twistors

Let (Ω, d) be a DG algebra, that is $\Omega = \bigoplus_{n \geq 0} \Omega^n$ is a graded algebra and $d: \Omega \rightarrow \Omega$ is a linear map with $d(\Omega^n) \subseteq \Omega^{n+1}$ for all $n \geq 0$, $d^2 = 0$ and $d(\omega\zeta) = d(\omega)\zeta + (-1)^{|\omega|}\omega d(\zeta)$ for all homogeneous ω and ζ , where $|\omega|$ is the degree of ω . The Fedosov product [12,15], given by

$$\omega \circ \zeta = \omega\zeta - (-1)^{|\omega|}d(\omega)d(\zeta), \quad (6.1)$$

for homogeneous ω and ζ , defines a new associative algebra structure on Ω . If we define the map

$$T: \Omega \otimes \Omega \rightarrow \Omega \otimes \Omega, \quad T(\omega \otimes \zeta) = \omega \otimes \zeta - (-1)^{|\omega|}d(\omega) \otimes d(\zeta), \quad (6.2)$$

then T satisfies (3.7) but fails to satisfy (3.5) and (3.6). However, the failure is only caused by some signs, so we were led to introduce a graded analogue of a twistor, which in turn leads us to the following much more general concept:

Proposition 6.1. *Let \mathcal{C} be a (strict) monoidal category, A an algebra in \mathcal{C} with multiplication μ and unit u , $T : A \otimes A \rightarrow A \otimes A$ a morphism in \mathcal{C} such that $T \circ (u \otimes A) = u \otimes A$ and $T \circ (A \otimes u) = A \otimes u$. Assume that there exist two morphisms $\tilde{T}_1, \tilde{T}_2 : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ in \mathcal{C} such that*

$$(A \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes A) = T \circ (A \otimes \mu), \quad (6.3)$$

$$(\mu \otimes A) \circ \tilde{T}_2 \circ (A \otimes T) = T \circ (\mu \otimes A), \quad (6.4)$$

$$\tilde{T}_1 \circ (T \otimes A) \circ (A \otimes T) = \tilde{T}_2 \circ (A \otimes T) \circ (T \otimes A). \quad (6.5)$$

Then $(A, \mu \circ T, u)$ is also an algebra in \mathcal{C} , denoted by A^T . The morphism T is called a pseudotwistor and the two morphisms \tilde{T}_1, \tilde{T}_2 are called the companions of T .

Proof. Obviously u is a unit for $(A, \mu \circ T)$, so we only check the associativity of $\mu \circ T$:

$$\begin{aligned} (\mu \circ T) \circ ((\mu \circ T) \otimes A) &= (\mu \circ T) \circ (\mu \otimes A) \circ (T \otimes A) \\ &\stackrel{(6.4)}{=} \mu \circ (\mu \otimes A) \circ \tilde{T}_2 \circ (A \otimes T) \circ (T \otimes A) \\ &\stackrel{(6.5)}{=} \mu \circ (\mu \otimes A) \circ \tilde{T}_1 \circ (T \otimes A) \circ (A \otimes T) \\ &= \mu \circ (A \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes A) \circ (A \otimes T) \\ &\stackrel{(6.3)}{=} \mu \circ T \circ (A \otimes \mu) \circ (A \otimes T) \\ &= (\mu \circ T) \circ (A \otimes (\mu \circ T)), \end{aligned}$$

finishing the proof. \square

Remark 6.2. Obviously, an ordinary twistor T is a pseudotwistor with companions $\tilde{T}_1 = \tilde{T}_2 = T_{13}$. Also, if $T : A \otimes A \rightarrow A \otimes A$ is a bijective R -matrix, one can easily check that T is a pseudotwistor, with companions $\tilde{T}_1 = T_{12} \circ T_{13} \circ T_{12}^{-1}$ and $\tilde{T}_2 = T_{23} \circ T_{13} \circ T_{23}^{-1}$.

A pseudotwistor may be thought of as some sort of analogue of a (Hopf) 2-cocycle, as suggested by the following examples (for which \mathcal{C} is the usual category of vector spaces):

Examples 6.3. Let H be a bialgebra and $F = F^1 \otimes F^2 = f^1 \otimes f^2 \in H \otimes H$ a Drinfeld twist, i.e. an invertible element (with inverse denoted by $F^{-1} = G^1 \otimes G^2$) such that

$$F^1 f_1^1 \otimes F^2 f_2^1 \otimes f^2 = f^1 \otimes F^1 f_1^2 \otimes F^2 f_2^2 \quad \text{and} \quad (\varepsilon \otimes H)(F) = (H \otimes \varepsilon)(F) = 1.$$

If A is a left H -module algebra, it is well known that the new product on A given by $a * b = (G^1 \cdot a)(G^2 \cdot b)$ is associative. This product is afforded by the map

$$T : A \otimes A \rightarrow A \otimes A, \quad T(a \otimes b) = G^1 \cdot a \otimes G^2 \cdot b,$$

and one may check that T is a pseudotwistor with companions \tilde{T}_1, \tilde{T}_2 given by the formulae

$$\begin{aligned}\tilde{T}_1(a \otimes b \otimes c) &= G^1 F^1 \cdot a \otimes G_1^2 F^2 \cdot b \otimes G_2^2 \cdot c, \\ \tilde{T}_2(a \otimes b \otimes c) &= G_1^1 \cdot a \otimes G_2^1 F^1 \cdot b \otimes G^2 F^2 \cdot c.\end{aligned}$$

Dually, let H be a bialgebra and $\sigma : H \otimes H \rightarrow k$ a normalized and convolution invertible left 2-cocycle (i.e. σ satisfies $\sigma(h_1, h'_1)\sigma(h_2 h'_2, h'') = \sigma(h'_1, h''_1)\sigma(h, h'_2 h''_2)$ for all $h, h', h'' \in H$). If A is a left H -comodule algebra with comodule structure $a \mapsto a_{(-1)} \otimes a_{(0)}$, one may consider the new associative product on A given by $a * b = \sigma(a_{(-1)}, b_{(-1)})a_{(0)}b_{(0)}$. This product is afforded by the map $T : A \otimes A \rightarrow A \otimes A$, $T(a \otimes b) = \sigma(a_{(-1)}, b_{(-1)})a_{(0)} \otimes b_{(0)}$, which is a pseudotwistor with companions \tilde{T}_1, \tilde{T}_2 given by the formulae

$$\begin{aligned}\tilde{T}_1(a \otimes b \otimes c) &= \sigma^{-1}(a_{(-1)1}, b_{(-1)1})\sigma(a_{(-1)2}, b_{(-1)2}c_{(-1)})a_{(0)} \otimes b_{(0)} \otimes c_{(0)}, \\ \tilde{T}_2(a \otimes b \otimes c) &= \sigma^{-1}(b_{(-1)1}, c_{(-1)1})\sigma(a_{(-1)}b_{(-1)2}, c_{(-1)2})a_{(0)} \otimes b_{(0)} \otimes c_{(0)}.\end{aligned}$$

In particular, for $A = H$, we obtain that the “twisted bialgebra” ${}_{\sigma}H$, with multiplication $a * b = \sigma(a_1, b_1)a_2b_2$, for all $a, b \in H$, is obtained as a deformation of H through the pseudotwistor $T(a \otimes b) = \sigma(a_1, b_1)a_2 \otimes b_2$ with companions $\tilde{T}_1(a \otimes b \otimes c) = \sigma^{-1}(a_1, b_1)\sigma(a_2, b_2c_1)a_3 \otimes b_3 \otimes c_2$ and $\tilde{T}_2(a \otimes b \otimes c) = \sigma^{-1}(b_1, c_1)\sigma(a_1b_2, c_2)a_2 \otimes b_3 \otimes c_3$, for all $a, b, c \in H$.

Lemma 6.4. *Let \mathcal{C} be a (strict) braided monoidal category with braiding c . Let V be an object in \mathcal{C} and $T : V \otimes V \rightarrow V \otimes V$ a morphism in \mathcal{C} . Then*

$$(V \otimes c_{V,V}) \circ (T \otimes V) \circ (V \otimes c_{V,V}^{-1}) = (c_{V,V}^{-1} \otimes V) \circ (V \otimes T) \circ (c_{V,V} \otimes V), \quad (6.6)$$

$$(V \otimes c_{V,V}^{-1}) \circ (T \otimes V) \circ (V \otimes c_{V,V}) = (c_{V,V} \otimes V) \circ (V \otimes T) \circ (c_{V,V}^{-1} \otimes V), \quad (6.7)$$

as morphisms $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ in \mathcal{C} . These two morphisms will be denoted by $\tilde{T}_1(c)$ and $\tilde{T}_2(c)$ and will be called the companions of T with respect to the braiding c . If $c_{V,V}^{-1} = c_{V,V}$ (for instance if \mathcal{C} is symmetric), the two companions coincide and will be simply denoted by $T_{13}(c)$.

Proof. The naturality of c implies $(V \otimes T) \circ c_{V \otimes V, V} = c_{V \otimes V, V} \circ (T \otimes V)$. Since c is a braiding, we have $c_{V \otimes V, V} = (c_{V,V} \otimes V) \circ (V \otimes c_{V,V})$, hence we obtain

$$(V \otimes T) \circ (c_{V,V} \otimes V) \circ (V \otimes c_{V,V}) = (c_{V,V} \otimes V) \circ (V \otimes c_{V,V}) \circ (T \otimes V).$$

By composing to the left with $c_{V,V}^{-1} \otimes V$ and to the right with $V \otimes c_{V,V}^{-1}$, we obtain the desired equality (6.6). Similarly one can check that (6.7) holds, too. \square

Definition 6.5. Let \mathcal{C} be a (strict) braided monoidal category, (A, μ, u) an algebra in \mathcal{C} and $T : A \otimes A \rightarrow A \otimes A$ a morphism in \mathcal{C} . Assume that

$$c_{A,A}^{-1} = c_{A,A}$$

(so we have the morphism $T_{13}(c)$ in \mathcal{C} as above). If T is a pseudotwistor with companions $\tilde{T}_1 = \tilde{T}_2 = T_{13}(c)$ and moreover $(T \otimes A) \circ (A \otimes T) = (A \otimes T) \circ (T \otimes A)$, we will call T a *braided twistor* for A in \mathcal{C} .

Consider now \mathcal{C} to be the category of \mathbb{Z}_2 -graded vector spaces, which is braided (even symmetric) with braiding given by $c(v \otimes w) = (-1)^{|v||w|} w \otimes v$, for v, w homogeneous elements. If (Ω, d) is a DG algebra, then Ω becomes a \mathbb{Z}_2 -graded algebra (i.e. an algebra in \mathcal{C}) by putting even components in degree zero and odd components in degree one. The map T given by (6.2) is obviously a morphism in \mathcal{C} , and using the above braiding one can see that the morphism $T_{13}(c)$ in \mathcal{C} is given by the formula $T_{13}(c)(\omega \otimes \zeta \otimes \eta) = \omega \otimes \zeta \otimes \eta - (-1)^{|\omega|+|\zeta|} d(\omega) \otimes \zeta \otimes d(\eta)$, for homogeneous ω, ζ, η (which is *different* from the ordinary T_{13}), and one can now check that T is a braided twistor for Ω in \mathcal{C} , and obviously Ω^T is just Ω endowed with the Fedosov product, regarded as a \mathbb{Z}_2 -graded algebra.

Theorem 6.6. *Let (A, μ, u) be an algebra in a (strict) monoidal category \mathcal{C} , let $T, R: A \otimes A \rightarrow A \otimes A$ be morphisms in \mathcal{C} , such that R is an isomorphism and a twisting map between A and itself. Consider the morphisms*

$$\tilde{T}_1(R) := (R^{-1} \otimes A) \circ (A \otimes T) \circ (R \otimes A), \quad (6.8)$$

$$\tilde{T}_2(R) := (A \otimes R^{-1}) \circ (T \otimes A) \circ (A \otimes R). \quad (6.9)$$

Define the morphism $P := R \circ T: A \otimes A \rightarrow A \otimes A$. Then:

(i) The relation (2.2) holds for P if and only if (6.3) holds for T , with $\tilde{T}_1(R)$ in place of \tilde{T}_1 .

(ii) The relation (2.3) holds for P if and only if (6.4) holds for T , with $\tilde{T}_2(R)$ in place of \tilde{T}_2 .

In particular, it follows that if T is a pseudotwistor for A with companions $\tilde{T}_1(R)$ and $\tilde{T}_2(R)$, then P is a twisting map between A and itself.

(iii) Conversely, assume that P is a twisting map and the following relations are satisfied:

$$(P \otimes A) \circ (A \otimes P) \circ (P \otimes A) = (A \otimes P) \circ (P \otimes A) \circ (A \otimes P), \quad (6.10)$$

$$(R \otimes A) \circ (A \otimes R) \circ (R \otimes A) = (A \otimes R) \circ (R \otimes A) \circ (A \otimes R), \quad (6.11)$$

$$(P \otimes A) \circ (A \otimes P) \circ (R \otimes A) = (A \otimes R) \circ (P \otimes A) \circ (A \otimes P), \quad (6.12)$$

$$(R \otimes A) \circ (A \otimes P) \circ (P \otimes A) = (A \otimes P) \circ (P \otimes A) \circ (A \otimes R). \quad (6.13)$$

Then T is a pseudotwistor for A with companions $\tilde{T}_1(R)$ and $\tilde{T}_2(R)$.

(iv) Assume that (iii) holds and moreover

$$(P \otimes A) \circ (A \otimes R) \circ (R \otimes A) = (A \otimes R) \circ (R \otimes A) \circ (A \otimes P), \quad (6.14)$$

$$(R \otimes A) \circ (A \otimes R) \circ (P \otimes A) = (A \otimes P) \circ (R \otimes A) \circ (A \otimes R). \quad (6.15)$$

Then R is also a twisting map between A^T and itself.

Proof. We prove (i), while (ii) is similar and left to the reader. Assume first that (2.2) holds for P . Then we can compute:

$$\begin{aligned} T \circ (A \otimes \mu) &= R^{-1} \circ P \circ (A \otimes \mu) \\ &\stackrel{(2.2)}{=} R^{-1} \circ (\mu \otimes A) \circ (A \otimes P) \circ (P \otimes A) \\ &= R^{-1} \circ (\mu \otimes A) \circ (A \otimes R) \circ (A \otimes T) \circ (R \otimes A) \circ (T \otimes A) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(2.2)}{=} (A \otimes \mu) \circ (R^{-1} \otimes A) \circ (A \otimes T) \circ (R \otimes A) \circ (T \otimes A) \\
&= (A \otimes \mu) \circ \tilde{T}_1(R) \circ (T \otimes A),
\end{aligned}$$

which is precisely the condition (6.3). Conversely, assuming that (6.3) holds, we compute:

$$\begin{aligned}
P \circ (A \otimes \mu) &= R \circ T \circ (A \otimes \mu) \\
&\stackrel{(6.3)}{=} R \circ (A \otimes \mu) \circ \tilde{T}_1(R) \circ (T \otimes A) \\
&= R \circ (A \otimes \mu) \circ (R^{-1} \otimes A) \circ (A \otimes T) \circ (R \otimes A) \circ (T \otimes A) \\
&\stackrel{(2.2)}{=} (\mu \otimes A) \circ (A \otimes R) \circ (A \otimes T) \circ (R \otimes A) \circ (T \otimes A) \\
&= (\mu \otimes A) \circ (A \otimes P) \circ (P \otimes A),
\end{aligned}$$

which is (2.2) for P . Now we prove (iii). By (i) and (ii), it is enough to check (6.5). We compute:

$$\begin{aligned}
\tilde{T}_1(R) \circ (T \otimes A) \circ (A \otimes T) &= (R^{-1} \otimes A) \circ (A \otimes T) \circ (R \otimes A) \circ (T \otimes A) \circ (A \otimes T) \\
&= (R^{-1} \otimes A) \circ (A \otimes R^{-1}) \circ (A \otimes P) \circ (P \otimes A) \\
&\quad \circ (A \otimes R^{-1}) \circ (A \otimes P) \\
&\stackrel{(6.13)}{=} (R^{-1} \otimes A) \circ (A \otimes R^{-1}) \circ (R^{-1} \otimes A) \circ (A \otimes P) \\
&\quad \circ (P \otimes A) \circ (A \otimes P) \\
&\stackrel{(6.10), (6.11)}{=} (A \otimes R^{-1}) \circ (R^{-1} \otimes A) \circ (A \otimes R^{-1}) \circ (P \otimes A) \\
&\quad \circ (A \otimes P) \circ (P \otimes A) \\
&\stackrel{(6.12)}{=} (A \otimes R^{-1}) \circ (R^{-1} \otimes A) \circ (P \otimes A) \circ (A \otimes P) \\
&\quad \circ (R^{-1} \otimes A) \circ (P \otimes A) \\
&= (A \otimes R^{-1}) \circ (T \otimes A) \circ (A \otimes R) \circ (A \otimes T) \circ (T \otimes A) \\
&= \tilde{T}_2(R) \circ (A \otimes T) \circ (T \otimes A).
\end{aligned}$$

(iv) We check (2.2) and leave (2.3) to the reader. We compute:

$$\begin{aligned}
R \circ (A \otimes \mu \circ T) &= R \circ (A \otimes \mu \circ R^{-1} \circ P) \\
&= R \circ (A \otimes \mu) \circ (A \otimes R^{-1}) \circ (A \otimes P) \\
&\stackrel{(2.2)}{=} (\mu \otimes A) \circ (A \otimes R) \circ (R \otimes A) \circ (A \otimes R^{-1}) \circ (A \otimes P) \\
&\stackrel{(6.11)}{=} (\mu \otimes A) \circ (R^{-1} \otimes A) \circ (A \otimes R) \circ (R \otimes A) \circ (A \otimes P) \\
&\stackrel{(6.14)}{=} (\mu \otimes A) \circ (R^{-1} \otimes A) \circ (P \otimes A) \circ (A \otimes R) \circ (R \otimes A) \\
&= (\mu \circ R^{-1} \circ P \otimes A) \circ (A \otimes R) \circ (R \otimes A) \\
&= (\mu \circ T \otimes A) \circ (A \otimes R) \circ (R \otimes A),
\end{aligned}$$

finishing the proof. \square

Our motivating example for Theorem 6.6 was provided by the theory of braided quantum groups, a concept introduced by M. Durdevich in [13] as a generalization of the usual braided groups (= Hopf algebras in braided categories, in Majid's terminology), which in turn contains as examples some important algebras such as braided and ordinary Clifford algebras, see [14]. If $G = (A, \mu, \Delta, \varepsilon, S, \sigma)$ is a braided quantum group (so σ is a bijective twisting map between A and itself) and $n \in \mathbb{Z}$, Durdevich defined some operators $\sigma_n : A \otimes A \rightarrow A \otimes A$ and proved that the maps $\mu_n : A \otimes A \rightarrow A$, $\mu_n = \mu \circ \sigma_n^{-1} \circ \sigma$, give new associative algebra structures on A (with the same unit). This result may be regarded as a consequence of Theorem 6.6. Indeed, for any n , the maps $R := \sigma_n$ and $P := \sigma$ satisfy the hypotheses of the theorem, hence the map $T := R^{-1} \circ P = \sigma_n^{-1} \circ \sigma$ is a pseudotwistor for A , giving rise to the associative multiplication μ_n .

More generally, if A is an algebra, Durdevich introduced the concept of *braid system* over A , as being a collection \mathcal{F} of bijective twisting maps between A and itself, satisfying the condition

$$(\alpha \otimes A) \circ (A \otimes \beta) \circ (\gamma \otimes A) = (A \otimes \gamma) \circ (\beta \otimes A) \circ (A \otimes \alpha), \quad \forall \alpha, \beta, \gamma \in \mathcal{F}.$$

If we take $\alpha, \beta \in \mathcal{F}$ and define $T : A \otimes A \rightarrow A \otimes A$, $T := \alpha^{-1} \circ \beta$, by Theorem 6.6 we obtain that T is a pseudotwistor for A , giving rise to a new associative multiplication on A .

We record the following two easy consequences of Theorem 6.6.

Corollary 6.7. *Let \mathcal{C} be a (strict) braided monoidal category with braiding c , (A, μ, u) an algebra in \mathcal{C} and $T : A \otimes A \rightarrow A \otimes A$ a morphism in \mathcal{C} ; assume also that $c_{A,A}^{-1} = c_{A,A}$. Define the morphism $R : A \otimes A \rightarrow A \otimes A$ by $R := c_{A,A} \circ T$. Then T satisfies the condition (6.3) (respectively (6.4)) with $T_{13}(c)$ in place of \tilde{T}_1 (respectively \tilde{T}_2) if and only if R satisfies (2.2) (respectively (2.3)). In particular, if T is a braided twistor for A in \mathcal{C} , then R is a twisting map between A and itself.*

Corollary 6.8. *Let \mathcal{C} be a (strict) braided monoidal category with braiding c and (A, μ, u) an algebra in \mathcal{C} . Then $T := c_{A,A}^2$ is a pseudotwistor for A in \mathcal{C} (this follows by taking $R = c_{A,A}^{-1}$ and $P = c_{A,A}$ in Theorem 6.6). In particular it follows that $(A, \mu \circ c_{A,A}^2, u)$ is a new algebra in \mathcal{C} .*

This algebra $(A, \mu \circ c_{A,A}^2, u)$ allows us to give an interpretation of the concept of *ribbon algebra* introduced by Akrami and Majid in [1], as an essential ingredient for constructing braided Hochschild and cyclic cohomology. Recall from [1] that a ribbon algebra in a braided monoidal category (\mathcal{C}, c) is an algebra (A, μ, u) in \mathcal{C} equipped with an isomorphism $\sigma : A \rightarrow A$ in \mathcal{C} such that $\mu \circ (\sigma \otimes \sigma) \circ c_{A,A}^2 = \sigma \circ \mu$ and $\sigma \circ u = u$ (such a σ is called a *ribbon automorphism* for A). The naturality of c implies

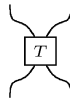
$$(\sigma \otimes \sigma) \circ c_{A,A}^2 = c_{A,A}^2 \circ (\sigma \otimes \sigma),$$

so the above relation may be written as

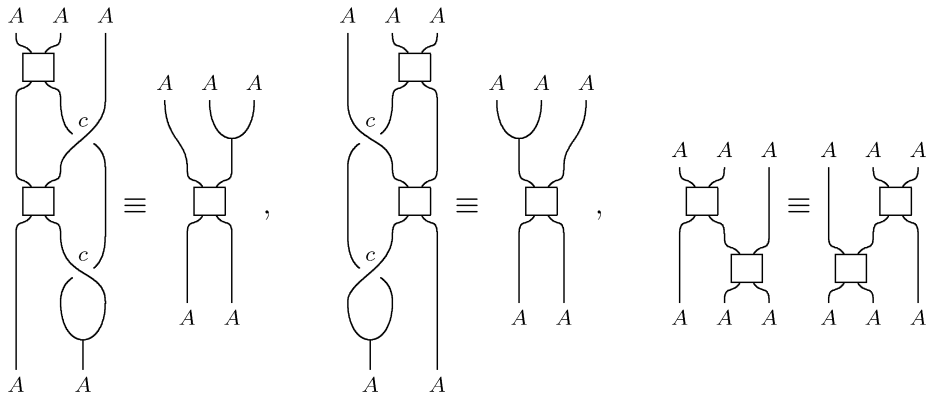
$$\mu \circ c_{A,A}^2 \circ (\sigma \otimes \sigma) = \sigma \circ \mu.$$

Hence, a ribbon automorphism for A is the same thing as an algebra isomorphism from (A, μ, u) to $(A, \mu \circ c_{A,A}^2, u)$.

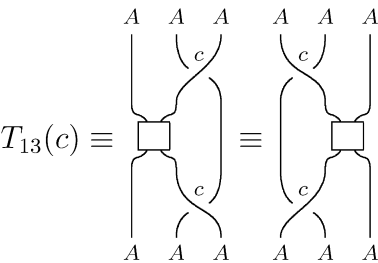
Let D be an algebra and T a twistor for D . We intend to lift T to the algebra ΩD of universal differential forms on D ; it will turn out that the natural way of doing this does *not* provide a twistor, but a braided twistor. In order to simplify the proof, we will use a braiding notation. Namely, we denote a braided twistor T for an algebra A in a braided monoidal category with braiding c satisfying $c_{A,A}^{-1} = c_{A,A}$ by



where we will omit the label T whenever there is no risk of confusion. With this notation, the conditions for T to be a braided twistor are written as:



It is also worth writing the two equivalent definitions of $T_{13}(c)$ using this notation, namely:



Let us consider now an algebra D together with a twistor $T : D \otimes D \rightarrow D \otimes D$. From Corollary 6.7 we know that the map $R := \tau \circ T$ is a twisting map between D and itself. But then, using Theorem 2.1, we may lift the twisting map R to a twisting map $\tilde{R} : \Omega D \otimes \Omega D \rightarrow \Omega D \otimes \Omega D$ between the algebra of universal differential forms ΩD and itself. Using again Corollary 6.7 in the category of graded vector spaces (with the graded flip τ_{gr} as a braiding) we obtain that the map $\tilde{T} : \Omega D \otimes \Omega D \rightarrow \Omega D \otimes \Omega D$ defined as $\tilde{T} := \tau_{\text{gr}} \circ \tilde{R}$ satisfies the conditions (6.3) and (6.4) with $\tilde{T}_1 = \tilde{T}_2 = T_{13}(\tau_{\text{gr}})$. Moreover, it is clear that $\tilde{T}^0 \equiv T$, since \tilde{R} extends R and the graded

flip coincides with the classical flip on degree 0 elements. Let us check that \tilde{T} also satisfies the condition

$$(\tilde{T} \otimes \Omega D) \circ (\Omega D \otimes \tilde{T}) = (\Omega D \otimes \tilde{T}) \circ (\tilde{T} \otimes \Omega D), \quad (6.16)$$

and hence \tilde{T} is a braided (graded) twistor for the algebra ΩD . In order to do this, we follow a standard procedure when dealing with differential calculi. First, as the restriction of \tilde{T} to $\Omega^0 D$ is a twistor, it satisfies the condition. Second, assume that the condition is satisfied for an element $\omega \otimes \eta \otimes \theta$ in $\Omega D \otimes \Omega D \otimes \Omega D$, and let us prove that it is also satisfied for $d\omega \otimes \eta \otimes \theta$, $\omega \otimes d\eta \otimes \theta$ and $\omega \otimes \eta \otimes d\theta$. First of all, realize that, for homogeneous ω , $\eta \in \Omega D$, we have

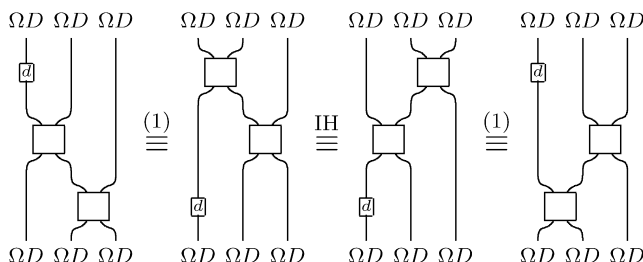
$$\tau_{\text{gr}}(\eta \otimes d\omega) = (-1)^{|d\omega||\eta|} d\omega \otimes \eta = (-1)^{(|\omega|+1)|\eta|} d\omega \otimes \eta = (\varepsilon \otimes d) \circ \tau_{\text{gr}}(\eta \otimes \omega), \quad (6.17)$$

where d and ε denote respectively the differential and the grading of ΩD . As a consequence of this equality and the compatibilities of \tilde{R} with the differential (cf. (2.7) and (2.8)), we realize immediately that the map \tilde{T} satisfies the following compatibility relations with the differential:

$$\tilde{T} \circ (d \otimes \Omega D) = (d \otimes \Omega D) \circ \tilde{T}, \quad (6.18)$$

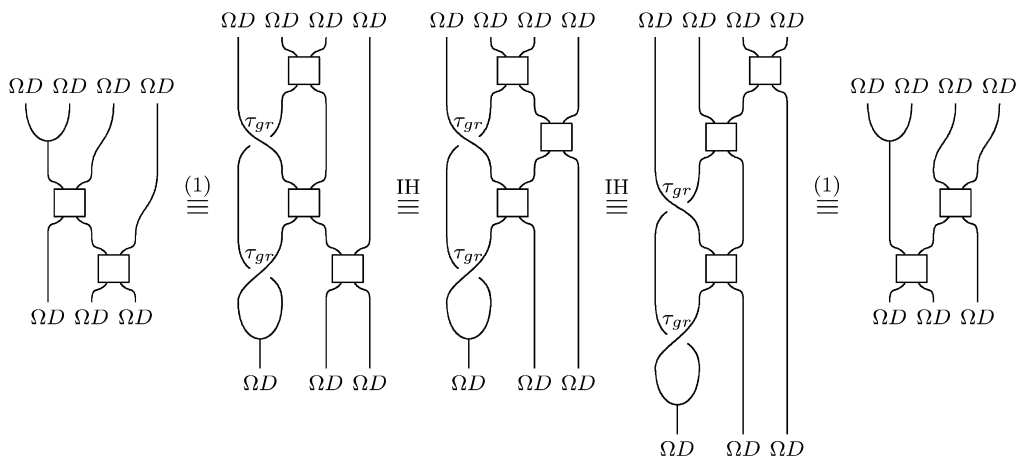
$$\tilde{T} \circ (\Omega D \otimes d) = (\Omega D \otimes d) \circ \tilde{T}. \quad (6.19)$$

Using braiding notation we have:



where in (1) we are using (6.18), and in the second equality we are using the induction hypothesis, and so the condition (6.16) for \tilde{T} behaves well under the differential in the first factor. The proof for the condition with the differential on the second or third factors is similar, and left to the reader.

Finally, we have to check that this condition also behaves well under products on any of the factors. For doing this, we need slightly stronger induction hypotheses. Namely, assume that we have $\omega_1, \omega_2, \eta, \theta$ such that the condition is satisfied for $\omega_i \otimes \eta' \otimes \theta'$, being η', θ' any elements in ΩD such that $|\eta'| \leq |\eta|$ and $|\theta'| \leq |\theta|$, i.e. we assume that the condition is true when we fix the ω_i 's and let the η' and θ' vary up to some degree bound, and let us prove that in this case the condition holds for $\omega_1 \omega_2 \otimes \eta' \otimes \theta'$. For this, take into account that \tilde{T} preserves the degree of homogeneous elements, since both \tilde{R} and τ_{gr} do. Now, we have



where in (1) we are using (6.3), in the equalities labeled with IH we are using our strengthened induction hypotheses. The desired result follows. Similar proofs exist when applying multiplication in the second or third factors. It is easy to see that, as a consequence of the properties we have just proved, we obtain that the map \tilde{T} is a braided (graded) twistor on the differential graded algebra ΩD . More concretely, we have proved the first part of the following result:

Theorem 6.9. *Let D be an algebra and $T : D \otimes D \rightarrow D \otimes D$ a twistor for D . Consider $R := \tau \circ T$, the twisting map associated to T . Let \tilde{R} be the extension of R to ΩD , then the map $\tilde{T} := \tau_{gr} \circ \tilde{R}$ is a braided (graded) twistor for ΩD . Moreover, the algebra $(\Omega D)^{\tilde{T}}$ is a differential graded algebra with differential d .*

Proof. The only part left to prove is that the map d is still a differential for the deformed algebra $(\Omega D)^{\tilde{T}}$, but this is an easy consequence of the fact that both the differential d and the grading ε commute with the twistor \tilde{T} . \square

The deformed algebra $(\Omega D)^{\tilde{T}}$ has, as the 0th degree component, the algebra D^T , and, whenever T is bijective, it is generated (as a graded differential algebra) by D^T , henceforth $(\Omega D)^{\tilde{T}}$ is a differential calculus over D^T . Thus, as a consequence of the Universal Property for the algebra of universal differential forms, we may conclude that $(\Omega D)^{\tilde{T}}$ is a quotient of the graded differential algebra $\Omega(D^T)$.

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